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HOLOMPRHIC VECTOR BUNDLES ON $\mathbf{C}^2\backslash\{0\}$

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ABSTRACT

Here we show the existence of a rank 2 holomorphic vector bundle E on $\mathbf{C}^2\backslash\{0\}$ without any holomorphic rank 1 subsheaf. Hence, contrary to the algebraic case, there are open subsets of dimension 2 Stein manifolds with holomorphic vector bundles which are not filtrable in any weak sense.

0. Introduction

By the Oka–Grauert principle for any Stein space W the holomorphic and the topological classification of complex vector bundles on W coincide ([G]). Several papers were devoted to the study of holomorphic vector bundles on complex analytic spaces. Here we study holomorphic vector bundles on $\mathbf{C}^2\backslash\{0\}$. Any such bundle is trivial as a topological or differentiable vector bundle (see, e.g., 1.1). A holomorphic vector bundle on $\mathbf{C}^2\backslash\{0\}$ is trivial if and only if it extends as a coherent analytic sheaf to $\mathbf{C}^2$ ([Se]). There are several non-trivial holomorphic vector bundles on $\mathbf{C}^2\backslash\{0\}$ (see [Se], p. 372, for a rank 1 example). By an extension theorem for coherent sheaves due to Siu–Trautmann and Guenot–Narasimhan ([S2], Prop. 7.1) the case "Stein two-dimensional manifold minus a discrete set" may be considered as the critical case. This explains our interest in the case of $\mathbf{C}^2\backslash\{0\}$. In algebraic geometry one can usually build a rank 2 vector bundle using only two line bundles and one zero-dimensional subscheme in the following way.

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Let E be a rank 2 holomorphic vector bundle on the punctured plane $\mathbf{C}^2\backslash\{0\}$. We will say that E is filtrable if there exists an exact sequence

$$0 \to L \to E \to M \otimes \mathbf{I}_Z \to 0 \tag{1}$$

with L and M holomorphic bundles on $\mathbf{C}^2\backslash\{0\}$ and Z a zero-dimensional analytic subspace of $\mathbf{C}^2\backslash\{0\}$. Since $\mathbf{C}^2\backslash\{0\}$ is smooth and two-dimensional, any rank 1 torsion free sheaf on $\mathbf{C}^2\backslash\{0\}$ is of the form $M \otimes \mathbf{I}_Z$ for some M and some zero-dimensional analytic subspace Z. The condition "E locally free" forces Z to be locally given by two equations, i.e., to be locally a complete intersection.

The aim of this paper is to prove the following result.

THEOREM 0.1: *There exists a non-filtrable rank 2 holomorphic vector bundle on* $\mathbf{C}^2\backslash\{0\}$.

The exponential sequence classifies all line bundles $\mathbf{C}^2\backslash\{0\}$: they are parametrized bijectively by the infinite-dimensional vector space $H^1(\mathbf{C}^2\backslash\{0\}, \mathbf{O}_{\mathbf{C}^2\backslash\{0\}})$ (see 1.1). The set of all rank 2 vector bundles E fitting in the exact sequence (1) may be considered known in terms of L, M and Z (see 1.7). Even the possible triples (L, M, Z) for which there is an exact sequence (1) with E locally free may be considered known (see 1.7). However, Theorem 0.1 shows that this information does not give the full picture.

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1. Proof of 0.1

Let $\mathbf{O}$ be the coherent sheaf of holomorphic functions on any open subset of $\mathbf{C}^2$.

(1.1) Here and in 1.2, ..., 1.8 we collect a few remarks on vector bundles on $\mathbf{C}^2\backslash\{0\}$ or the punctured bidisk $\Delta_2\backslash\{0\}$ or $\Delta \times \mathbf{C}\{(0,0)\}$. Hence here X will denote either $\mathbf{C}^2\backslash\{0\}$ or $\Delta_2\backslash\{0\}$ or $\Delta \times \mathbf{C}\backslash\{(0,0)\}$ and set $X^+ := \mathbf{C}^2$ in the first case, $X^+ := \Delta_2$ in the second case and $X^+ := \Delta \times \mathbf{C}$ in the third case. Hence X is homotopically equivalent to S^3. Every holomorphic vector bundle on X is topologically trivial; indeed taking a hermitian metric, the structural group of any such bundle may be reduced to $U(n)$; by [Hu], Cor. 8.4, the $U(n)$-bundles on S^3 are classified by $\pi_2(U(n))$ and this homotopy group is trivial by [Hu], 12.4. Let F be a holomorphic vector bundle on X. Since every holomorphic vector bundle on $\mathbf{C}^2$ or on Δ_2 or on $\Delta \times \mathbf{C}$ is trivial by the Oka–Grauert principle

([G]), F extends across $\{0\}$ as a holomorphic vector bundle if and only if it is trivial; since a reflexive sheaf on a two-dimensional complex manifold is locally free ([Ha], Cor. 1.2 and Prop. 1.9), F extends over $\{0\}$ as a holomorphic vector bundle if and only if it extends as a coherent sheaf; this is the case if and only if $F|(U|\{0\})$ is spanned by its global sections for some small neighborhood U of $\{0\}$ ([Se], Th. 1); this is not always the case, even for line bundles ([Se], p. 372). Now we study the case $\mathrm{rank}(F) = 1$. From the exponential exact sequence we obtain $H^1(X, \mathbf{O}^*) \cong H^1(X, \mathbf{O})$. $H^1(X, \mathbf{O})$ is an infinite-dimensional complex vector space (see [GR], pp. 130–131, for its complete description in the case $X = \mathbf{C}^2\backslash\{0\}$; even if $X = \Delta_2\backslash\{0\}$ the cohomology group $H^1(X, \mathbf{O})$ is an inifinite-dimensional vector space for the following reason; let U be an open subset of $\mathbf{C}^2$; we have $H^2(U, \mathbf{O}_U) = 0$ ([S1]), while $H^1(U, \mathbf{O}_U)$ has infinite dimension if U is not Stein ([Co], Th. 1.1, and [B], Cor. on p. 657)). Thus $\mathrm{Pic}(X) \cong H^1(X, \mathbf{O}^*)$ is an infinite-dimensional vector space. In particular, if $L \in \mathrm{Pic}(X)$ and $L^{\otimes t} \cong \mathbf{O}$ for some integer $t \geq 2$, then $L \cong \mathbf{O}$.

LEMMA 1.2: *Let L be a holomorphic line bundle on X. L is trivial if and only if $H^0(X, L) \neq \{0\}$.*

Proof: If $L \cong \mathbf{O}$, then $H^0(X, L) \cong H^0(X, \mathbf{O})$ is infinite-dimensional. Assume $H^0(X, L) \neq \{0\}$ and take $s \in H^0(X, L)$, $L \neq 0$. Since X is smooth and connected, s defines an exact sequence

$$0 \to \mathbf{O} \to L \to \mathbf{O}_D \to 0 \tag{2}$$

with either $D = \emptyset$ (i.e., s nowhere vanishing) or D an effective Cartier divisor on X. In the first case $L \cong \mathbf{O}$ by (2). Assume $D \neq \emptyset$. Since D has pure dimension 1, its closure D^- in X^+ defines an effective Cartier divisor D^- on X^+ ([S2], Th. 2.15). Since $H^2(X^+, \mathbf{Z}) = H^1(X^+, \mathbf{O}) = 0$, D^- is principal, i.e., there exists a holomorphic function f on X^+ vanishing (counting multiplicity) exactly on D^-. Thus f induces a local trivialization around $\{0\}$ of L. Hence L is trivial.

COROLLARY 1.3: *Let E be a rank 2 vector bundle on X fitting in (1) with $L \cong M \cong \mathbf{O}$. Then every rank 1 line bundle contained in E is trivial.*

Proof: Take $R \in \mathrm{Pic}(x)$ such that there is an injection j: $R \to E$ as coherent sheaves. If $j(R)$ is contained in the subsheaf $L \cong \mathbf{O}$ of E defined by (1), then R^* is trivial by Lemma 1.2 and hence R is trivial. If $j(R)$ is not contained in L, then j induces an injection $R \to M \otimes \mathbf{I}_Z$ and, in particular, an inclusion of R into $M \cong \mathbf{O}$. By Lemma 1.2, R^* is trivial and hence we conclude.

Remark 1.4: Every Cartier divisor is the difference of two effective Cartier divisors. Hence Lemma 1.2 implies that $\mathbf{O}$ is the unique holomorphic line bundle on X associated to a Cartier divisor.

(1.5) Fix any $L \in \mathrm{Pic}(X)$ and set $H := \{z \in X : z_2 = 0\}$. H is a principal divisor and hence $L(-H) \cong L$. We have $L|H \cong \mathbf{O}_H$ because every line bundle on H is trivial. If $L \cong \mathbf{O}$ we have $H^1(X, L)$ is an infinite-dimensional $\mathbf{C}$-vector space by the last part of 1.1. If L is not trivial we have $H^0(X, L) = \{0\}$ by 1.2. Hence if L is not trivial, from the exact sequence

$$0 \to L(-H) \to L \to L \to 0 \tag{3}$$

we obtain that $H^1(X, L)$ is an infinite-dimensional vector space.

(1.6) Fix holomorphic line bundles L, M on X. Consider the set $\mathrm{Ext}^1(X; M, L)$ of all extension classes

$$0 \to L \to E \to M \to 0 \tag{4}$$

on X. $\mathrm{Ext}^1(X; M, L)$ is an infinite-dimensional $\mathbf{C}$-vector space. Any coherent sheaf fitting in (4) is a rank 2 holomorphic vector bundle.

(1.7) Fix holomorphic line bundles L, M on X and a (perhaps not reduced) zero-dimensional analytic subspace Z of X such that Z is locally a complete intersection. Consider the set $\mathrm{Ext}^1(X; M \otimes \mathbf{I}_Z, L)$ of all extension classes (1) on X. Since at each point of Z_{red} the analytic space Z is locally a complete intersection of two local equations, we have $\mathrm{Ext}^1(M \otimes \mathbf{I}_Z, L) \cong L \otimes M^* \otimes \mathbf{O}_Z \cong \mathbf{O}_Z$. By the local-to-global spectral sequence for the Ext-functor we have $\mathrm{Ext}^1(X; M \otimes \mathbf{I}_Z, L) \cong H^1(X, L \otimes M^*) \oplus \mathbf{O}_Z$. Using this decomposition of the global Ext-functor we see that the extension (4) defines a locally free sheaf E if the corresponding extension has a component generating $\mathbf{O}_Z$ ([Ca]). By 1.5 and the last part of 1.1 (case $L \cong M$) the cohomology group $H^1(X, L \otimes M^*)$ is an infinite-dimensional vector space.

(1.8) Fix any rank 2 vector bundle E on X fitting in an exact sequence (4). Let R be a holomorphic line bundle on X such that there exists a holomorphic map u: $R \to E$ with $u(R)$ not contained in L. Then u induces a non-zero map v: $R \to M \otimes \mathbf{I}_Z$. By 1.2 we obtain $R \cong M$ and that Z is contained in an effective Cartier divisor, D, on X. Such D extends across $\{0\}$ ([S2], Th. 2.15); call D^+ its closure in X^+. Lemma 1.9 below shows that this is a restrictive condition on Z.

LEMMA 1.9 (J. Winkelmann): *Fix complex numbers α_1 and α_2 with $0 < |\alpha_1| \leq |\alpha_2| < 1$ and set $Z := \{(\alpha_1^k, \alpha_2^k)\}_{k\in\mathbf{Z}} \subset \mathbf{C}^2\backslash\{0\}$. There exists a holomorphic function on $\mathbf{C}^2\backslash\{0\}$ vanishing on Z if and only if there are positive integers a, b with $\alpha_1^a = \alpha_2^b$.*

Proof: If $\alpha_1^a = \alpha_2^b$, just take $f(z_1, z_2) = z_1^a - z_2^b$. Now assume that there are no such integers a, b. Every holomorphic function on $\mathbf{C}^2\backslash\{0\}$ extends to $\mathbf{C}^2$ and $Z \cup \{0\}$ is the closure of Z in $\mathbf{C}^2$. Assume the existence of a holomorphic function f on $\mathbf{C}^2$ with $f(Z) = \{0\}$. Choose a real number c such that $|\alpha_1|^c = |\alpha_2|$. For each monomial $z_1^k z_2^x$ we associate a weight $k + cx$. This is motivated by the observation that $|(\alpha_1^n)^k(\alpha_2^n)^x| = |\alpha_1^n|^{(k+cx)}$. We have $f(z_1, z_2) = \sum_{r\in\mathbf{R}} f_r$, where for every real number r, f_r is the sum of all monomials of weight r contained in the power series expansion of f. Thus $|f_r(\alpha_1^n, \alpha_2^n)| = K_r|\alpha_1|^{nr}$ for some real constant K_r. We claim that $K_r = 0$ for every r. If the claim is not true, there is a smallest real number r' with $K_{r'} > 0$. If n is large enough, the term $f_{r'}$ would dominate all other terms at points near $(0, 0)$ and then $K_{r'} > 0$ would contradict the assumption $f(Z) = \{0\}$. Thus $K_r = 0$ for every r, i.e., $f_r(Z) = \{0\}$ for every r. If f_r is the sum of at most one monomial, $f_r(Z) = \{0\}$ implies $f_r \equiv 0$. If c is not rational, each f_r consists of at most one monomial. Thus we may assume c rational, say $c = p/q$ for some positive integers p, q. Thus $|\alpha_1|^c = |\alpha_2|$ means $|\alpha_1|^p = |\alpha_2|$. Let s, t be complex numbers with $t^q = a$ and $t^p s = b$. For each r we have $0 = f_r(\alpha_1^n, \alpha_2^n) = \sum_{k+cx=r} a_{kx}\alpha_1^{kn}\alpha_2^{bxn} = \sum_{qk+px=qr} a_{kx}t^{qr}s^{xn} = \sum_{qk+px=qr^a kx} t^{qr}s^{xn} = t^{qr}(\sum_{qk+px=qr} a^{kx}s^{xn})$. Hence $\sum_{qk+px} a^{kx}s^{xn} = 0$ for every integer n. By the usual results on Vandermonde determinants, this may happen only if either all a_{kx} vanish or some of the s^x coincide. In the first case we have $f \equiv 0$. In the second case $s^m = 1$ for some integer $m > 0$ and hence $\alpha_1^{pm} = \alpha_2^{qm}$.

Proof of Theorem 0.1: Fix complex numbers α_1 and α_2 with $0 < |\alpha_1| \leq |\alpha_2| < 1$ and define $g \in \mathrm{Aut}(\mathbf{C}^2\backslash\{0\})$ by $g(z_1, z_2) := (\alpha_1 z_1, \alpha_2 z_2)$. Let π: $\mathbf{C}^2\backslash\{0\} \to Y := \mathbf{C}^2\backslash\{0\}/G$ be the quotient map by the group $G \cong \mathbf{Z} \subsetneq \mathrm{Aut}(\mathbf{C}^2\backslash\{0\})$ generated by g. G acts freely on $\mathbf{C}^2\backslash\{0\}$. It is known (see, e.g., [Ko2], p. 695) that Y is a complex compact non-algebraic surface (a so-called primary Hopf surface). Y has algebraic dimension $a(Y) = 1$ if and only if there are positive integers a, b with $\alpha_1^a = \alpha_2^b$ ([Ko2], Th. 31); if there is no such pair of positive integers (a, b), then Y has algebraic dimension $a(Y) = 0$. From now on we assume that α_1 and α_2 are such that $a(Y) = 0$. Y is homeomorphic to $S^1 \times S^3$ ([Ko1]). Y belongs to Kodaira's class VII_0, i.e., we have $b_1(Y) = 1$, $h^1(Y, \mathbf{O}_Y) = 1$ and $h^0(Y, \omega_Y) = 0$. Thus $\chi(\mathbf{O}_Y) = 0$ and $\omega_Y \cdot \omega_Y = 0$. Fix an integer $c_2 \geq 3$ and let Z be a general subset of Y with $\mathrm{card}(Z) = c_2$. We claim the existence of a rank 2 holomorphic vector bundle F on Y fitting in an exact sequence

$$0 \to \mathbf{O}_Y \to F \to \mathbf{I}_Z \to 0. \tag{5}$$

Since $h^0(Y, \omega_Y) = 0$, for every subset Z' of Z with $\mathrm{card}(Z') = \mathrm{card}(Z) - 1$, we have $h^0(Y, \omega_Y \otimes \mathbf{I}_{Z'}) = 0$ and hence the Cayley–Bacharach condition is satisfied ([Ca]), proving the claim. Set $A := \pi^*(F)$ and $T := \pi^{-1}(Z)$. Thus A is a rank 2 holomorphic vector bundle on $\mathbf{C}^2\backslash\{0\}$ and we have the following exact sequence:

$$0 \to \mathbf{O} \to A \to \mathbf{I}_T \to 0. \tag{6}$$

By Lemma 1.9 the closure $T \cup \{0\}$ of T in $\mathbf{C}^2$ is not an analytic set. Since $H^0(\mathbf{C}^2\backslash\{0\}, \mathbf{O}) = H^0(\mathbf{C}^2, \mathbf{O})$ the assumption implies that A is not trivial and cannot be extended to $\mathbf{C}^2$. By Corollary 1.3 every line bundle contained in A is trivial. By Lemma 1.9, $\mathbf{O}$ is the subsheaf of A spanned by $H^0(\mathbf{C}^2\backslash\{0\}, A)$. We have $c_1(F) \cong \mathbf{O}_Y$. With the notation of [BL], p. 9, we call $m(2, \mathbf{O}_Y)$ a certain integer on Y; we assume $c_2 \geq \max\{m(2, \mathbf{O}_Y), 5\}$. The proof of [BL], Th. 5.2 (case $a(Y) = 0$) shows the existence of a flat family $\{F_z\}_{z\in\Delta}$ of holomorphic vector bundles on Y with special fiber $F_0 \cong F$ and with F_z irreducible for $z \in (\Delta\backslash\{0\})$. Set $E_z := \pi^*(F_z)$. Hence $\{E_z\}_{z\in\Delta}$ is a flat family of holomorphic vector bundles on $\mathbf{C}^2\backslash\{0\}$ with $E_0 \cong A$. We want to prove that E_z is not filtrable for general $z \in \Delta$. We claim that E is not trivial for general z and we will give two proofs of this claim. The family $\{E_z\}_{z\in\Delta}$ defines a vector bundle, $\mathbf{A}$, over $(\mathbf{C}^2\backslash\{0\}) \times \Delta$; if, for a thick set of $z \in \Delta$, $\mathbf{A}|(\mathbf{C}^2\backslash\{0\}) \times \{z\}$ extends over $\{0\} \times \{z\}$, then $\mathbf{A}$ extends to a coherent sheaf $\mathbf{B}$ over $\mathbf{C}^2 \times \Delta$ ([S2], Prop. 7.1); hence $\mathbf{B}|\mathbf{C}^2 \times \{0\}$ extends E_z as a coherent sheaf; thus E_0 extends across $\{0\}$ as a vector bundle ([Se]), contradiction. Alternatively, if E_z is trivial, then it has 2 linearly independent G-invariant holomorphic sections and hence $h^0(Y, F_z) \geq 2$, contradiction. Hence if for a general z the vector bundle E_z is filtrable, then E_z has a unique rank 1 subsheaf, L_z, as abstract holomorphic line bundle; since $\det(E_z)$ is trivial, then by 1.3, L_z is unique as saturated subsheaf of E_z if L_z is not trivial. First assume L_z not trivial. Notice that each E_z comes from Y and hence it is G-invariant. By its uniqueness L_z is G-invariant. Since G acts freely, this implies that E_z descends to a rank 1 coherent subsheaf of E_z. Since E_z is not filtrable, we obtain a contradiction. Now assume L_z trivial. Thus E_z fits in an exact sequence

$$0 \to \mathbf{O} \to E_z \to \mathbf{I}_{Z(z)} \to 0. \tag{7}$$

Set $B(E_z) := \{P \in \mathbf{C}^2\backslash\{0\} : E_z \text{ is not spanned at } P\}$. $B(E_z)$ is an analytic G-invariant subset of $\mathbf{C}^2\backslash\{0\}$. If no non-zero section of $\mathbf{I}_{Z(z)}$ lifts to a section of

E_z, then the inclusion of $\mathbf{O}$ in E_z given by (7) is unique and hence G-invariant, contradicting the non-filtrability of F_z. Hence $B(E_z) \neq \mathbf{C}^2\backslash\{0\}$. Since $B(E_z)$ is G-invariant and Y has no curve, $B(E_z)$ cannot contain any one-dimensional analytic component. Hence either $B(E_z)$ is discrete or $B(E_z) = \emptyset$. Since $a(Y) = 0$, then by Lemma 1.9, $\mathbf{C}^2\backslash\{0\}$ has no discrete non-empty G-invariant analytic subset. Thus $B(E_z) = \emptyset$, i.e., E_z is spanned. By [Se], Th. 1, E_z extends across $\{0\}$, i.e., E_z is trivial, contradiction.

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